# Submanifold Differential Operators in $\mathcal{D}$-Module Theory II: Generalized Weierstrass and Frenet-Serret Relations as Dirac Equations

S. MATSUTANI
8-21-1 Higashi-Linkan Sagamihara 228-0811 JAPAN

Abstract.

This article is one of a series of papers. For this decade, the Dirac operator on a submanifold has been studied as a restriction of the Dirac operator in $n$-dimensional euclidean space $\mathbb{E}^n$ to a surface or a space curve as physical models. These Dirac operators are identified with operators of the Frenet-Serret relation for a space curve case and of the generalized Weierstrass relation for a conformal surface case and completely represent the submanifolds. For example, the analytic index of Dirac operator of a space curve is identified with its writhing number. As another example, the operator determinants of the Dirac operators are closely related to invariances of the immersed objects, such as Euler-Bernoulli and Willmore functionals for a space curve and a conformal surface respectively. In this article, we will give mathematical construction of the Dirac operator by means of $\mathcal{D}$-module and reformulate my recent results mathematically.

**MCS Codes:** 32C38, 34L40, 35Q40

**Key Words:** Weierstrass Relation, Frenet-Serret Relation, Dirac operator, Submanifold, $\mathcal{D}$-Module

## §1. Introduction

In an earlier article in this series [I], we showed the construction of the submanifold Schrödinger operator in terms of $\mathcal{D}$-module theory. In this article, we will apply the scheme to the spin bundle to construct the Dirac operator of submanifold. We will call the previous article [I] and its proposition or definition and reference like (I-2.1) and [I-Mat2], which means proposition or definition 2-1 and reference [Mat2] in [I] respectively.

Applying the quantum mechanical scheme [I and its references] to Dirac operators for a restricted particle along a low-dimensional submanifold in $n$-dimensional euclidean space $\mathbb{E}^n$, we obtained natural Dirac operators on curves in $\mathbb{E}^n$ ($n \geq 2$) [Mat1-3, 5, 6, 8, 10, MT] and on conformal surfaces in $\mathbb{E}^n$ ($n = 3, 4$) [BJ, Mat11, 14, 16]. In this decade, I have been studying these Dirac operators and investigating their properties. From physical point of view, I showed that they exhibit the symmetry of corresponding submanifolds and found a non-trivial extension of Atiyah-Singer type index theorem to submanifold [Mat2,6]. The Dirac operators of curves in $\mathbb{E}^n$ ($n \geq 2$) are related to the Frenet-Serret relations and are identified with the Lax operators of $(1+1)$-dimensional soliton equations, *e.g.*, modified Korteweg-de Vries equation [MT, Mat2,8], nonlinear Schrödinger equation [Mat1, 3, 6], complex modified Korteweg-de Vries equation [Mat15], and so on [Mat3]. The Dirac operators on conformal surfaces in $\mathbb{E}^n$ ($n = 3, 4$) are concerned with the generalized Weierstrass equation representing a surface [Mat11, 14,16] and also identified with the Lax operators of modified Novikov-Veselov (MNV) equations [Ko1, 2, T1, 2]. The generalized Weierstrass equation is very interesting from the viewpoint of immersion geometry and differential geometry [B, Ke, Ko1, 2, TK, T1, 2, PP].

For example, for a case that $S$ is a conformal surface immersed in $\mathbb{E}^3$, the generalized Weierstrass equation is given as the zero mode of the Dirac operator,

$$\nabla\!\!\!\!/^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^3} = \sqrt{-1}\rho^{1/2}\begin{pmatrix} p & \partial \\ \bar\partial & p \end{pmatrix}\rho^{-1/2},$$

where $(z, \bar{z})$ is a complex parameterization of $S$, $p$ is $\rho^{1/2}H$ when the volume element of $S$ is given by $\mathrm{dvol} = \rho dz d\bar{z}$ and $H$ is the mean curvature. This Dirac operator was obtained by various methods, *e.g.*, direct computations [Ko1, 2], quotation algebraic method [Pe, PP] spin bundle consideration [Fr] and quantum mechanical scheme [BJ, Mat11,14,16].

Similarly we can obtain the generalized Weierstrass equation of $\mathbb{E}^4$ case [Ko2, PP, Mat16]. The quantum mechanical scheme [Mat16] is one of the easiest approaches. Thus in order that mathematicians understand the recent progress of quantum mechanical approaches, I believe that the construction scheme should be translated to mathematical language and be more precise.

In 1991, I applied the Jensen and Koppe [I-JK] and da Costa [I-dC] scheme of submanifold quantum mechanics to the Dirac operator over a plane curve as a first example and investigated it for this decade [MT, Mat2, 5, 10]. Here we will summary those. Along the scheme described in the introduction of [I], we obtain the Dirac operator on a real analytic curve in $\mathbb{E}^2$ as [MT, Mat2, 6, 9],

$$\nabla\!\!\!\!/^{\mathrm{Dir}}_{C\hookrightarrow\mathbb{E}^2} = \begin{pmatrix} v & -\sqrt{-1}\partial_s \\ -\sqrt{-1}\partial_s & v \end{pmatrix},$$

where $s$ is the arclength of the curve, $v := k/2$ and $k$ is the curvature of the curve. The Dirac equation as a zero mode of this Dirac operator,

$$\nabla\!\!\!\!/^{\mathrm{Dir}}_{C\hookrightarrow\mathbb{E}^2}\psi = 0,$$

is essentially equivalent with the Frenet-Serret relation. By considering the eigenvalue equations,

$$\nabla\!\!\!\!/^{\mathrm{Dir}}_{C\hookrightarrow\mathbb{E}^2}\psi = \lambda\psi,$$

and deformation of the curve preserving $\lambda$, we obtain the MKdV equation,

$$\partial_t v + 6v^2\partial_1^3 v + \partial_1^3 v = 0,$$

where $t$ is deformation parameter which can be physically interpreted as the real time of the curve approximately [MT, Mat10] and an adiabatic parameter in the fermionic system [MT, Mat2, 8, 12, 17].

Furthermore operator determinant of $\nabla\!\!\!\!/^{\mathrm{Dir}}_{C\hookrightarrow\mathbb{E}^2}$ is the invariant of the system, viz the Euler-Bernoulli functional

$$\log\det \nabla\!\!\!\!/^{\mathrm{Dir}}_{C\hookrightarrow\mathbb{E}^2} = \frac{1}{2\pi}\int ds\ k^2(s),$$

and its index is given as

$$\mathrm{ind}\,\nabla\!\!\!\!/^{\mathrm{Dir}}_{C\hookrightarrow\mathbb{E}^2} = w_{C\hookrightarrow\mathbb{E}^2},$$

where $w_{C\hookrightarrow\mathbb{E}^2}$ is the writhing number of the curve [MT, Mat2, 6, 8, 14]. Moreover the moduli of the immersion curve is classified by the Dirac operator [Mat12, 17]. Hence the interpretation of the Dirac operator means the investigation of the curve itself. Hence I believe that my Dirac operator is very important in geometry.

This procedure, as I conjectured in [Mat2], can be extended to a surface case and we can obtain the generalized Weierstrass relation, Novikov-Veselov equation and Willmore functional.

Here I will comment upon my philosophical idea on these studies. When I started these studies, I did not have any proper language to express my motivation. However now I can express my motivation why I started these.

In algebraic number theory, we encounter the Fermat problems whether there exists an integer pair $(x, y) \in \mathbb{Z}^2$ satisfied with the relation $x^2 + y^2 = p$ or $x^2 + 3y^2 = p$ for a given prime number $p$ [Coh]. These are solved in quadratic form theory and studied by quadratic integer. For the case of $x^2 + y^2 = p$, if $p \equiv 1$ modulo $4\mathbb{Z}$, there exists such integer pair $(x, y)$. Due to the Legendre symbol $\left(\dfrac{-1}{p}\right) = 1$ for $p \equiv 1$ modulo $4\mathbb{Z}$, there is a number $z$ such that $z^2 + 1 \equiv 0$ modulo $p\mathbb{Z}$. Hence $p$ is not prime number if we consider it in $\mathbb{Z}[\sqrt{-1}]$; $p\mathbb{Z}[\sqrt{-1}]$ is not prime ideal in $\mathbb{Z}[\sqrt{-1}]$ and must be decomposed. (Since the equation $x^2 + y^2 \equiv 0$ modulo $p\mathbb{Z}$ is decomposed to $(x + \sqrt{-1}y)(x - \sqrt{-1}y) \equiv 0$ modulo $p\mathbb{Z}[\sqrt{-1}]$, it means that for $p \equiv 1$ modulo $4\mathbb{Z}$, there exists $(x, y) \in \mathbb{Z}^2$ such that $(x + \sqrt{-1}y)(x - \sqrt{-1}y) \equiv 0$ modulo $p\mathbb{Z}[\sqrt{-1}]$.) Similarly we will consider $x^2 + 3y^2 = p$ as $(x + \sqrt{-3}y)(x - \sqrt{-3}y) \equiv 0$ modulo $p\mathbb{Z}[\sqrt{-3}]$. These Abel extensions decompose the set of prime ideals $\mathrm{Spec}(\mathbb{Z})$, which is a first step of class field theory.

Here we will note that $\sqrt{-1}$ and $\sqrt{-3}$ are generators of orientation (charity or complex conjugate) of $\mathbb{Z}[\sqrt{-1}] \subset \mathbb{C}$ or $\mathbb{Z}[\sqrt{-3}] \subset \mathbb{C}$ and this extension $\mathbb{Z}[\sqrt{-1}]$ leads us to the algebraic number theory, ideal theory and arithmetic geometrical theory. Further we will note that in algebraic number theory, sets of prime ideals ( *e.g.*, $\mathrm{Spec}(\mathbb{Z})$, $\mathrm{Spec}(\mathbb{Z}[\sqrt{-1}])$, $\mathrm{Spec}(\mathbb{Z}[\sqrt{-3}])$, and so on) are more important than prime numbers themselves.

Jensen and Koppe and da Costa [I-dC, I-JK] found the natural quadratic operator for a case of plane curve;

$$\Delta_{C\hookrightarrow\mathbb{E}^2} := \Delta_C + \frac{1}{2}k^2,$$

where $\Delta_C$ is the Beltrami-Laplace operator of curve and $\frac{1}{2}k^2$ is integrand of the Euler-Bernoulli functional. The Euler-Bernoulli functional and $\Delta_C$ are invariant of the system and hermite. Let the set of the spectrum of $\Delta_{C\hookrightarrow\mathbb{E}^2}$ be denoted by $\mathrm{Spect}(\Delta_{C\hookrightarrow\mathbb{E}^2})$. Then up to $\mathrm{Spect}(\Delta_{C\hookrightarrow\mathbb{E}^2})$, $\Delta_{C\hookrightarrow\mathbb{E}^2} \equiv 0$ in a certain sense. By adding a generator of orientation $k$ in $\mathbb{E}^2$, ($k(s)$ changes its sign depending upon the orientation), we will encounter the Dirac operator $\not\nabla^{\mathrm{Dir}}_{C\hookrightarrow\mathbb{E}^2}$ [MT], which is not hermite,

$$\frac{1}{2}\mathrm{tr}_{2\times 2}((\not\nabla^{\mathrm{Dir}}_{C\hookrightarrow\mathbb{E}^2})^2) = \Delta_{C\hookrightarrow\mathbb{E}^2},$$

and $\mathrm{Spect}(\Delta_{C\hookrightarrow\mathbb{E}^2})$ is quadratically decomposed to the spectrum $\mathrm{Spect}(\not\nabla^{\mathrm{Dir}}_{C\hookrightarrow\mathbb{E}^2})$ of $\not\nabla^{\mathrm{Dir}}_{C\hookrightarrow\mathbb{E}^2}$, in a certain sense. This is resemble with the quadratic form theory [Coh].

Of course, we might encounter "fail of unique factorization" in this problem. As prime ideal must be more important than prime number in algebraic number theory, this problem should be also expressed by a module theory. Thus I have chosen $\mathcal{D}$-module theory as mathematical expression of my theory. Even though in noncommutative ring, prime ideal becomes nonsense, I wish to construct a quadratic form theory in this problem.

In other words, my motivation is arithmeticalization of quantum physics or differential geometry and harmonic map theory. I believe that the operator space of the Dirac operator plays the same role as $\mathbb{Z}[\sqrt{-1}]$ and $\mathbb{Z}[\sqrt{-3}]$. This aspect can be regarded as another aspect of the program of Eichler [E].

Recently number theory becomes gemetorized [E, Fa] and physics and geometry also becomes arithmeticalized [Mat18 and references therein]. In future, both might be unified. I hope that my approach might shed a new light on this way.

Contents are as follows. §2 quickly reviews the Clifford algebra, the spin group and its spinor representation, which are well-established [ABS, BGV, G, Y]. The purpose of §2 is to show notations in this article. In §3, using well-known results on the Clifford, spin and spinor bundle [ABS, BGV, G, Y], we will follow the argument of Mallios [Mal1, 2] and formally construct sheaves of spin,

Clifford module and so on. In §4, we will define the Dirac operator in a submanifold and give theorems.

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## §2. Spinor Group

Let us review the spinor group [BGV, Ch, G].

**Definition 2.1.**

*Let $V$ be a real vector with a positive definite inner product $(,)$.*

(1) *We will define an equivalent relation $\sim_{\mathrm{CLIFF}}$ by $v \cdot v + (v, v) = 0$ for $v \in V$.*

(2) *The universal algebra generated by $V$ with $\mathbb{R}$ coefficient is denoted by $\mathrm{UNV}(V)$.*

(3) *The Clifford algebra $\mathrm{CLIFF}(V)$ is defined by*

$$\mathrm{CLIFF}(V) = \mathrm{UNV}(V)/ \sim_{\mathrm{CLIFF}} .$$

**Lemma 2.2.**

*The relation $v \cdot v + (v, v) = 0$ is given by the basis*

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

*where $\delta_{ij}$ is Kronecker delta symbol.*

*Proof.* First we will consider the case $v = e_i$. Then we have $e_i \cdot e_i = -1$. Next let $v = \sum v^i e_i$. The relation is $v \cdot v = \sum_{i,j} v^i v^j e_i \cdot e_j + \sum_i (v^i)^2 = 0$ and reduces to $e_i e_j + e_j e_i = 0$ for $i \neq j$.

Let we will show the representation of $\mathrm{CLIFF}(V)$ using endomorphism of the exterior algebra $\Lambda V$ of $V$, $\mathrm{END}(\Lambda V)$.

**Definition 2.3.**

*Let an orthonormal basis of $V$ be $\{e_j\}_{i=1,\cdots,n}$.*

(1) *Let $\mathrm{ext} : V \to \mathrm{END}(\Lambda V)$ be exterior multiplication on the left. Its action is given as,*

$$\mathrm{ext}(e_1)(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \begin{cases} e_1 \wedge e_{i_1} \wedge \cdots \wedge e_{i_p}, & \text{if } i_1 \neq 1 \\ 0, & \text{otherwise} \end{cases} .$$

(2) *Let $\mathrm{int} : V \to \mathrm{END}(\Lambda V)$ be interior multiplication through the adjoint map $* : V \to V^* := \mathrm{Hom}(V, \mathbb{R})$. Its action is given as*

$$\mathrm{int}(e_1)(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \begin{cases} e_{i_2} \wedge \cdots \wedge e_{i_p}, & \text{if } i_1 = 1 \\ 0, & \text{otherwise} \end{cases} .$$

(3) *Let* cliff : $V \to \mathrm{END}(\Lambda V)$ *be clifford multiplication,*

$$\mathrm{cliff}(v) := \mathrm{ext}(v) - \mathrm{int}(v).$$

**Lemma 2.4.**

(1) *There is a ring endomorphism by extension of the action* cliff : $V \to \mathrm{END}(\Lambda V)$

$$\mathrm{cliff} : \mathrm{CLIFF}(V) \to \mathrm{END}(\Lambda V).$$

(2) $\Lambda V$ *is* $\mathrm{CLIFF}(V)$*-module.*

(3) *There is a vector space isomorphism as* $\mathrm{CLIFF}(V)$*-module,*

$$\mathfrak{cliff} : \mathrm{CLIFF}(V) \to \Lambda V, \quad (a \mapsto \mathfrak{cliff}(a) := \mathrm{cliff}(a)(1)).$$

*Let us express* $\mathfrak{c} := (\mathfrak{cliff})^{-1}$.

*Proof.* Noting $\mathrm{ext}(v)^2 = \mathrm{int}(v)^2 \equiv 0$ for $v \in V$, we have a relation,

$$(\mathrm{cliff}(v))^2 = -(\mathrm{ext}(v)\mathrm{int}(v) + \mathrm{int}(v)\mathrm{ext}(v)) = -(v,v)I.$$

First we will consider cliff : $\mathrm{UNV}(V) \to \mathrm{END}(V)$ as for $w_1, w_2 \in \mathrm{UNV}(V)$, $\mathrm{cliff}(w_1 + w_2) = \mathrm{cliff}(w_1) + \mathrm{cliff}(w_2)$. and $w_1, w_2 \in \mathrm{UNV}(V)$, $\mathrm{cliff}(wv) := \mathrm{cliff}(w)\mathrm{cliff}(v)$. Then above relation is consist with $\sim_{\mathrm{CLIFF}}$ and cliff : $\mathrm{CLIFF}(V) \to \mathrm{END}(\Lambda V)$ is well-defined. Others are easy. □

**Definition 2.5.**

*We will decompose* $\mathrm{CLIFF}(V)$ *as a module to submodule* $\mathrm{CLIFF}_+(V)$ *consisting of multiplication of even elements of* $V$ *and submodule* $\mathrm{CLIFF}_-(V)$ *consisting of multiplication of odd elements of* $V$*;*

$$\mathrm{CLIFF}(V) = \mathrm{CLIFF}_+(V) \oplus \mathrm{CLIFF}_-(V).$$

**Definition 2.6 (Spin Group).**

*Let* $*$ *be involution of* $\mathrm{CLIFF}(V) \to \mathrm{CLIFF}(V)$ *as* $(e_{i_1} \cdots e_{i_p})^* = (e_{i_p} \cdots e_{i_1})$.

*We will define the spin group,*

$$\mathrm{SPIN}(V) = \{g \in \mathrm{CLIFF}_+(V) \mid g^* g = 1,\ g^* v g \in V,\ \forall v \in V\}.$$

*For* $V = \mathbb{R}^n$ *case, we will denote it* $\mathrm{SPIN}(n)$.

**Lemma 2.7.**

*When define the degree of* $\mathrm{CLIFF}(V)$ *by a number of elements of* $V$, $\deg(e_1, \cdots, e_n) = n$ *and graded Lie bracket,*

$$[A, B]_\pm = AB - (-)^{\deg(A)\cdot\deg(B)} BA,$$

$\mathrm{CLIFF}_+(V) := \mathfrak{c}(\Lambda^2 V)$ *with the graded Lie bracket* $[,]_\pm$ *is a Lie algebra, which is isomorphic to the Lie algebra* $\mathrm{so}(V)$*, and* $\mathrm{SPIN}(V)$ *is a compact Lie group associated with the Lie algebra.*

*Proof.* (p.105 in [BGV]) The isomorphism $\tau : \mathfrak{c}(\Lambda^2 V) \to \mathrm{so}(V)$ is obtained by letting $a \in \mathfrak{c}(\Lambda^2 V)$ acting on $\mathfrak{c}(\Lambda^1 V) \approx V$ by the adjoint action: $\tau(a) \cdot v = [a, v]_\pm \equiv av + va$. Then we obtain $\tau(a) : \mathfrak{c}(\Lambda^1 V) \to \mathfrak{c}(\Lambda^1 V)$. Hence : $\mathfrak{c}(\Lambda^2 V)$ is Lie subalgebra of $\mathrm{gl}(V)$. Due to the Jacobi identity,

$$[[a, v]_\pm, w]_\pm + [[v, a]_\pm, w]_\pm + [[w, v]_\pm, a]_\pm = 0, \quad \text{for } v, w \in V,$$

and $[[w, v]_\pm, a]_\pm = 0$, we obtain $(\tau(a) \cdot v, w) = -(v, \tau(a) \cdot w)$. Hence $\tau(a)$ is isomorphism. From the construction of $\tau$, $\mathfrak{c}(\Lambda^2 V)$ is the Lie algebra associated with the spin group $\mathrm{SPIN}(V)$. □

For a matrix $A \in \mathrm{so}(V)$, the Clifford element is given as,

$$\tau^{-1}(A) = \frac{1}{2} \sum_{i,j} (Ae_i, e_j)\mathrm{cliff}(e_i)\mathrm{cliff}(e_j).$$

**Proposition 2.8.**
$\dim V > 1$, *there is a homomorphism* $\tau$,

$$\tau : \mathrm{SPIN}(V) \to \mathrm{SO}(V), \quad (\tau(g)v = gvg^*),$$

*which is 2-fold covering map.*

*Proof.* (p.106-p.107 in [BGV]) $\tau$ is locally isomorphic. Let $g$ be an element of kernel of $\tau$ or $\tau(g) = 1$. Then for all $v \in \mathrm{CLIFF}(\Lambda^1 V) \equiv V$, $[g, v]_\pm = 0$. From $\mathrm{cliff}([v, g]_\pm) = -2\mathrm{int}(v)\mathrm{cliff}(g) = 0$, $g$ must be scalar. From the definition $g \cdot g^* = 1$, the kernel of $\tau$ is $g = \pm 1$. □

**Corollary 2.9.** $\pm 1 \in \mathbb{Z}_2$ *is a center of* $\mathrm{SPIN}(V)$.

**Definition 2.10 (Spinor $\mathbb{C}$ Group).** [G p.186]

$$\mathrm{SPIN}^{\mathbb{C}}(V) := \mathrm{SPIN}(V) \times \mathbb{C}^\times / (\{(\epsilon, \epsilon), \epsilon = \pm 1\}.$$

*Here we will identify the* $\pm 1$ *as a center of* $\mathrm{SPIN}(V)$ *and* $\pm 1 \in \mathbb{C}^\times$. *We will denote it* $\mathrm{SPIN}^{\mathbb{C}}(n)$ *for* $V = \mathbb{R}^n$.

Then next is obvious.

**Proposition 2.11.**

$$1 \to \mathbb{C}^\times \to \mathrm{SPIN}^{\mathbb{C}}(n) \to \mathrm{SO}(n) \to 1,$$

$$1 \to \mathrm{SPIN}(n) \to \mathrm{SPIN}^{\mathbb{C}}(n) \to \mathbb{C}^\times \to 1.$$

Next we will consider the representation of the spin group.

**Proposition 2.12.**
*For* $n \in \mathbb{N}$, *these relations hold;*
(1)

$$\mathrm{CLIFF}(\mathbb{R}^{n+2}) \otimes_{\mathbb{R}} \mathbb{C} \approx \mathrm{CLIFF}(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathrm{END}(\mathbb{C}^2).$$

(2)

$$\mathrm{CLIFF}(\mathbb{R}^{2n}) \otimes_{\mathbb{R}} \mathbb{C} \approx \mathrm{END}(\mathbb{C}^{2^n}).$$

$$\mathrm{CLIFF}(\mathbb{R}^{2n+1}) \otimes_{\mathbb{R}} \mathbb{C} \approx \mathrm{END}(\mathbb{C}^{2^n}) \oplus \mathrm{END}(\mathbb{C}^{2^n}).$$

*Proof.* Let us define a Pauli matrices $\sigma_a \in \mathrm{END}(\mathbb{C}^2)$ $(a = 0, 1, 2, 3)$,

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -\sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

For an orthonormal basis $\{e_j\}_{i=1,\cdots,n+2}$ of $\mathbb{R}^{n+2}$, we will introduce a linear map

$$\mathbb{R}^{n+2} \to \mathrm{CLIFF}(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathrm{END}(\mathbb{C}^2),$$

$$e_j \mapsto e_j \otimes \sigma_3, \quad e_{n+1} \mapsto 1 \otimes \sigma_2, \quad e_{n+2} \mapsto 1 \otimes \sigma_1.$$

We can extend this relation to $\mathrm{UNV}(\mathbb{R}^{n+2}) \to \mathrm{CLIFF}(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathrm{END}(\mathbb{C}^2)$. Since

$$\sigma_1\sigma_2 = \sqrt{-1}\sigma_3, \quad \sigma_2\sigma_3 = \sqrt{-1}\sigma_1, \quad \sigma_3\sigma_1 = \sqrt{-1}\sigma_2,$$

$$e_j \otimes \sigma_3 \cdot e_k \otimes \sigma_3 = e_j e_k \otimes \sigma_0.$$

For $a \in \mathrm{CLIFF}(\mathbb{R}^n)$, following relations hold,

$$\begin{aligned} a \cdot e_{n+1} &\mapsto a \otimes (\sigma_3)^{\deg a}\sigma_2, \\ a \cdot e_{n+2} &\mapsto a \otimes (\sigma_3)^{\deg a}\sigma_1, \\ a\cdot &\mapsto a \otimes (\sigma_3)^{\deg a}, \\ a \cdot e_{n+1}e_{n+2} &\mapsto a \otimes (\sigma_3)^{\deg a+1}. \end{aligned}$$

So (1) is proved. On (2), we ill note the relation, as a first step,

$$\mathrm{END}(\mathbb{C}^n) \otimes \mathrm{END}(\mathbb{C}^2) = \mathrm{END}(\mathbb{C}^{2n}).$$

For even case, we can prove it using the relation

$$\mathrm{CLIFF}(\mathbb{R}^2) \otimes_{\mathbb{R}} \mathbb{C} \approx \mathrm{END}(\mathbb{C}^2),$$

which is given by Pauli matrices. For odd case,

$$\mathrm{CLIFF}(\mathbb{R}^1) \otimes_{\mathbb{R}} \mathbb{C} \approx \mathrm{END}(\mathbb{C}) \oplus \mathrm{END}(\mathbb{C}) \approx \mathbb{C} \oplus \mathbb{C}.$$

$\mathrm{CLIFF}(\mathbb{R}^1) \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{span}_{\mathbb{C}}\{1, e\} \to \mathrm{span}_{\mathbb{C}}\left\{\frac{1+e}{2}, \frac{1-e}{2}\right\}$. Inductively we can prove it. $\square$

**Proposition 2.13 (Spinor Representation of Clifford Algebra).**

(1) $\mathrm{CLIFF}(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$ *as a* $\mathrm{CLIFF}(V)$*-module is expressed as*

$$\mathrm{CLIFF}(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C} = \sum_{i=1}^{2^{[\frac{n}{2}]}} \Psi_i,$$

*where* $\Psi_i \approx \Psi$ *for* $i = 1, \cdots, 2^{[\frac{n}{2}]}$.

(2) *if* $n$ *is even,* $\Psi$ *is irreducible as a* $\mathrm{CLIFF}(\mathbb{R}^n)$*-module.*

(3) *if* $n$ *is odd,* $\Psi \approx \Psi_+ \oplus \Psi_-$ *and* $\Psi_\pm$ *are irreducible.*

*Proof.* (1) is obvious from the previous proposition. If we introduce a chiral operator and involution

$$\Gamma := e_1 e_2 \cdots e_n,$$

we have the relation

$$\Gamma e_j = (-1)^{n-1} e_j \Gamma.$$

Hence $\Gamma$ is a center of $\mathrm{CLIFF}(V)$ if $n$ is odd. Thus $\Gamma v = v\Gamma$ for $v \in \mathrm{CLIFF}_+(\mathbb{R}^n)$ for any $n \in \mathbb{N}$.

Finally, we will give only results.

**Proposition 2.14 (Spinor Representation of Spin Group).**

(1) $\mathrm{CLIFF}(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C}$ *as a* $\mathrm{SPIN}(n)$*-module, called as spinor-representation, is expressed as*

$$\mathrm{CLIFF}(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C} = \sum_{i=1}^{2^{[\frac{n}{2}]}} \Psi_i,$$

*where* $\Psi_i \approx \Psi_1$ *for* $i = 2, \cdots, 2^{[\frac{n}{2}]}$ *and* $\Psi_i$ *is a representation of* $\mathrm{SPIN}(V)$*.*

(2) *if* $n$ *is even,* $\Psi$ *is is reduced to* $\Psi \approx \Psi_+ \oplus \Psi_-$ *as* $\mathrm{SPIN}(V)$*-module and* $\Psi_+ \not\approx \Psi_-$

(3) *if* $n$ *is odd,* $\Psi \approx \Psi_+ \oplus \Psi_-$ *and* $\Psi_+ \approx \Psi_-$*.*

## §3. Spinor Sheaf

As we wish to construct a theory in the framework of the sheaf theory in the differential geometry and $\mathcal{D}$-module, we will go on to study the sheaf theory for the spinor geometry. As far as I know, there is no reference of the studies of spinor bundle from sheaf theoretical point of view. However though Atiyah, Bott and Shapiro did not used terminology of sheaf in the study of Clifford module [ABS], their studies can be easily translated to language of sheaf theory. In this article, we will translate the theory of spinor bundle in [ABS, BGV, G, Y] to language of sheaf theory.

Let $(M, \mathcal{C}^\omega_M)$ be a $n$-dimensional real analytic algebraized manifold without singularity and $\mathcal{C}^\omega_M$ is its structure sheaf of real analytic functions over $M$ and it has the Riemannian metric $(\Theta_M, \mathfrak{g}_M)$, where $\Theta_M$ is a tangent sheaf [I, Mal1,2].

Generally a sheaf $\mathcal{S}$ (of sets) over a topological space $X$ is characterized by a triple $(\mathcal{S}, \pi, X)$ because it is defined so that there is a local homeomorphism $\pi : \mathcal{S} \to X$ [Mal1]. However as we did in [I], we will also write it $\mathcal{S} := (\mathcal{S}, \pi, X)$ for abbreviation. Further we will write the set of (local) sections of $\mathcal{S}$ over an open set $U$ of $X$ as $\mathcal{S}(U)$ or $\Gamma(\mathcal{S}, U)$. Using the category equivalence between category of sheaves and category of complete presheaves (due to theorem 13.1 in [Mal1]), we will mix them.

We will go on to use the notations and definitions in the part I [I] unless we give notions. The sheaves, $\mathbb{R}_M$, $\mathbb{R}^{>0}_M$, $\mathbb{Z}_M$, $\mathbb{Z}_{2M}$, $\mathcal{H}om(\cdot,\cdot)$, $\Theta_M$, $\mathcal{I}(\cdot)$, $\mathcal{GL}(n, \mathcal{C}^\omega_M)$, $\Omega_M$, $\sqrt{\Omega_M}$, $\sqrt{\mathcal{C}^\omega_M}$ $\mathcal{D}_M$ and so on, are defined in [I]. We also employ the Einstein convention as in [I].

We will set up the group sheaf as a translation of a principal bundle as a subsheaf of $\mathcal{GL}(n, \mathcal{C}^\omega_M)$ using the results of Mallios [Mal1,2].

**Definition 3.1.** ( Principle Sheaves $\mathcal{SO}(n, \mathcal{C}^\omega_M)$ and $\mathcal{O}(n, \mathcal{C}^\omega_M)$) [Mal1]

(1) $SO(M)$ $(O(M))$ *is a principle bundle of* $\mathrm{SO}(n)$ $(\mathrm{O}(n))$ *over* $n$*-dimensional manifold* $M$*,*

$$\begin{array}{ccc} \mathrm{SO}(n) & \longrightarrow & SO(M) \\ & & \Big\downarrow \\ & & M. \end{array}$$

(2) *$\mathcal{SO}(n, \mathcal{C}^\omega_M)$ ($\mathcal{O}(n, \mathcal{C}^\omega_M)$) is a sheaf for real analytic sections of $SO(M)$ ($O(M)$).*

For an open set $U$ of $M$, there is a local coordinate system $\{x^i, \partial_i\}_{1\leq i\leq n}$ and $\{e_j\}_{j=1,\cdots,n} \in \Theta_M$ and $\{e^j\}_{j=1,\cdots,n} \in \Omega^1_M$ is an orthonormal set satisfying the relations $< e^i, e_j > = \delta^i_j$. We will fix the notations in this section.

**Definition 3.2.** (The Levi-Civita Connection) [Mal1]
*The Levi-Civita connection ${\nabla^{\mathrm{LC}}_M}_\theta \in \Gamma(U, \mathcal{H}om_{\mathcal{C}^\omega_M}(\Theta_M, \mathcal{I}(\Theta_M)))$ is defined as an integrable connection over the Riemannian tangent sheaf $(\Theta_M, \mathfrak{g}_M)$ for an open set $U$ of $M$ and $\theta \in \Theta_M(U)$ if it holds the following relations.*

(1) *For sections $\theta_1, \theta_2 \in \Theta_M(U)$,*

$$ {\nabla^{\mathrm{LC}}_M}_{\theta_1}\theta_2 - {\nabla^{\mathrm{LC}}_M}_{\theta_2}\theta_1 = [\theta_1, \theta_2]. $$

(2) *When we define $\nabla^{\mathrm{LC}}_M \in \Gamma(U, \mathcal{H}om_{\mathbb{R}_M}(\Theta_M, \Theta_M \otimes \Omega^1_M))$ as $\nabla^{\mathrm{LC}}_M := {\nabla^{\mathrm{LC}}_M}_{e_j} \otimes e^j$ using the local orthonormal set,*

$$ d\mathfrak{g}_M(\theta_1, \theta_2) = \mathfrak{g}_M(\nabla^{\mathrm{LC}}_M\theta_1, \theta_2) + \mathfrak{g}_M(\theta_1, \nabla^{\mathrm{LC}}_M\theta_2). $$

**Lemma 3.3.**
*The tangent bundle $TM$ of $M$ can be regarded as an associate bundle for a representation of $\rho : \mathrm{O}(n) \to \mathbb{R}^n$,*

$$ TM = O(M) \times_{\mathrm{O}(n)} \mathbb{R}^n. $$

*Proof.* it is obvious [BGV]. □

**Corollary 3.4.**
*There is an action of $\mathcal{SO}(n, \mathcal{C}^\omega_M)$ ($\mathcal{O}(n, \mathcal{C}^\omega_M)$) on $\Theta_M$: $\mathcal{SO}(n, \mathcal{C}^\omega_M)$ ($\mathcal{O}(n, \mathcal{C}^\omega_M)$) is a subset of $\mathcal{I}(\Theta_M)$.*

When the Levi-Civita connection ${\nabla^{\mathrm{LC}}_M}_\theta$ in $\Gamma(U, \mathcal{H}om_{\mathcal{C}^\omega_M}(\Theta_M, \mathcal{I}(\Theta_M)))$ belongs to $\Gamma(U, \mathcal{H}om_{\mathcal{C}^\omega_M}(\Theta_M, \mathcal{SO}(n, \mathcal{C}^\omega_M)))$ ($\Gamma(U, \mathcal{H}om_{\mathcal{C}^\omega_M}(\Theta_M, \mathcal{O}(n, \mathcal{C}^\omega_M))))$. Then we call it the Levi-Civita connection of $\mathcal{SO}(n, \mathcal{C}^\omega_M)$ ($\mathcal{O}(n, \mathcal{C}^\omega_M)$).

We also use the correspondence of lemma 3.3 and will introduce the Clifford sheaf.

**Definition 3.5 (Clifford Module).** [ABS, Y]

(1) *We will define a Clifford bundle $CLIFF(M)$ as a frame bundle of principle $O(M)$ bundle induced from the representation $\mathrm{O}(n) \to \mathrm{CLIFF}(\mathbb{R}^n)$.*

$$ CLIFF(M) = O(M) \times_{\mathrm{O}(n)} \mathrm{CLIFF}(\mathbb{R}^n). $$

*For $p \in M$, its fiber is $\mathrm{CLIFF}(T^*_pM)$.*

(2) *We will define a sheaf $\mathcal{C}\mathit{liff}_M$, which is generated by a set of real analytic sections of $CLIFF(M)$ over an open set $U$ of $M$.*

**Proposition 3.6 (Clifford Module).**

(1) *When we will regard* $\mathcal{C}\mathit{liff}_M$ *as a Clifford module, we have decomposition for an open set* $U \in M$*,*

$$\mathcal{C}\mathit{liff}_M(U) = \mathcal{C}\mathit{liff}_{M+}(U) \oplus \mathcal{C}\mathit{liff}_{M-}(U),$$

*corresponding to* $\mathrm{CLIFF}_+(\mathbb{R}^n)$ *and* $\mathrm{CLIFF}_-(\mathbb{R}^n)$ *respectively. Similarly we have the subbundles of* $\mathit{CLIFF}(M)$ *given by the Whitney sum,*

$$\mathit{CLIFF}(M) = \mathit{CLIFF}_+(M) \oplus \mathit{CLIFF}_-(M).$$

(2) *A sheaf morphism* $\mathfrak{c} : \Omega_M \to \mathcal{C}\mathit{liff}_M$ *can defined by local relation* $\mathfrak{c} : \Omega_M(U) \to \mathcal{C}\mathit{liff}_M(U)$*.*

*Proof.* [ABS, Y] Since they are locally defined, they are obvious form argument of §2. □

**Definition 3.7 (Clifford Connection).** [ABS p.117, Y]
*An integrable connection* $\nabla^{\mathrm{Clf}}_{M\,\theta} \in \Gamma(U, \mathrm{Hom}_{\mathcal{C}^\omega_M}(\Theta_M, \mathcal{I}(\mathcal{C}\mathit{liff}_M))$ *for an open set* $U \subset M$ *and* $\theta \in \Theta_M(U)$ *is called Clifford connection if it is satisfied with the following relation: for* $\mathfrak{c}(a) \in \mathcal{C}\mathit{liff}_M(U)$

$$[\nabla^{\mathrm{Clf}}_M, \mathfrak{c}(a)] = \mathfrak{c}(\nabla^{\mathrm{LC}}_M a),$$

*where* $\nabla^{\mathrm{LC}}_M$ *is a Levi-Civita connection.*

**Lemma 3.8 (Spin Group Sheaf $\mathcal{S}pin(n, \mathcal{C}^\omega_M)$).**
*Let us define a* $\mathit{SPIN}(M)$ *as a subbundle* $\mathit{CLIFF}_+(M)$*, which can be locally defined by the map* $\tau : \mathit{SPIN}(M)|_U \to \mathit{SO}(M)|_U$ *of proposition 2.8. Similarly we will define a* $\mathcal{S}pin(n, \mathcal{C}^\omega_M)$ *as a group sheaf generated by analytic sections of* $\mathit{SPIN}(M)$*, which is called spin group sheaf.*

**Proposition 3.9 (Bockstein exact sequence).**
*For the exact sequence of sheaf,*

$$1_M \to \mathbb{Z}_{2\,M} \to \mathcal{S}pin(n, \mathcal{C}^\omega_M) \to \mathcal{O}(n, \mathcal{C}^\omega_M) \to 1_M,$$

*there is the Bockstein exact sequence of Čech cohomology,*

$$0 \to \check{\mathrm{H}}^1(\mathbb{Z}_{2\,M}) \to \check{\mathrm{H}}^1(\mathcal{S}pin(n, \mathcal{C}^\omega_M)) \to \check{\mathrm{H}}^1(\mathcal{O}(n, \mathcal{C}^\omega_M)) \to \check{\mathrm{H}}^2(\mathbb{Z}_{2\,M}).$$

*Here* $\check{\mathrm{H}}^2(\mathbb{Z}_{2\,M})$ *is called second Stiefel-Whitney class.*

*Proof.* It is obvious from the general theory of exact sequence of sheaf. □

**Definition 3.10 (Spin $\mathbb{C}$ Group Sheaf $\mathcal{S}pin^{\mathbb{C}}$).**
*The spin* $\mathbb{C}$ *group sheaf* $\mathcal{S}pin^{\mathbb{C}}$ *is defined by local sections over an open set* $U \subset M$

$$\Gamma(U, \mathcal{S}pin^{\mathbb{C}}(n, \mathcal{C}^\omega_M)) := \Gamma(U, \mathcal{S}pin(n, \mathcal{C}^\omega_M)) \oplus \mathbb{C}^\times_M(U)/(\{(\epsilon, \epsilon), \epsilon = \pm 1\}).$$

*where* $\epsilon$ *is a local section of* $\mathbb{Z}_{2\,M}(U)$*.*

**Proposition 3.11.**

*There are sheaf short exact sequences of group sheaves,*

$$1_M \to \mathbb{C}_M^\times \to \mathcal{S}pin^{\mathbb{C}}(n, \mathcal{C}_M^\omega) \to \mathcal{SO}(n, \mathcal{C}_M^\omega) \to 1_M,$$

$$0 \to \mathbb{Z}_M \to \mathbb{C}_M \overset{\exp}{\to} \mathbb{C}_M^\times \to 0.$$

*Proof.* From 2.11, there is an exact sequence of their germs and first one is obvious. Second is following. For germ $g_x \in \mathbb{C}_M^\times$, there exists germ $f_x \in \mathbb{C}_M$ such that $g_x = \exp(f_x)$ because $f_x \equiv \log g_x$ modulo $2\pi\sqrt{-1}\mathbb{Z}$. $\square$

**Definition 3.12.** (Spin $\mathbb{C}$ structure)

*We say that n-dimensional manifold $M$ has $\mathcal{S}pin^{\mathbb{C}}$ structure if there is its associated principle bundle,*

$$\begin{array}{ccc} \mathrm{SPIN}^{\mathbb{C}}(\mathbb{R}^n) & \longrightarrow & SPIN^{\mathbb{C}}(M) \\ & & \downarrow \\ & & M. \end{array}$$

*and the cotangent bundle $T^*M$ of $M$ is equivalent with $SPIN^{\mathbb{C}}(M) \times_{\mathrm{SPIN}^{\mathbb{C}}(\mathbb{R}^n)} \mathbb{R}^n$.*

**Proposition 3.13.** (Spin $\mathbb{C}$ structure) [Y]

*There is a $\mathcal{S}pin^{\mathbb{C}}$ structure for a manifold $M$, if and only if $\alpha \in \check{\mathrm{H}}^2(\mathbb{Z}_M)$ such that*

$$w_2(M) = \alpha \quad \textit{modulo} \;\; 2\mathbb{Z}_M.$$

*Proof.* Let us introduce a correspondence of germs

$$\tau_{1x} : \mathcal{S}pin^{\mathbb{C}}(n, \mathcal{C}_M^\omega)_x \ni [\sigma, \lambda] \mapsto \tau(\sigma) \in \mathcal{SO}(n, \mathcal{C}_M^\omega)_x, \quad \tau_{2x} : \mathcal{S}pin^{\mathbb{C}}(n, \mathcal{C}_M^\omega)_x \ni [\sigma, \lambda] \mapsto \lambda \in \mathbb{C}_{Mx}^\times,$$

First we consider the necessary condition. Suppose that $M$ has $\mathcal{S}pin^{\mathbb{C}}$ structure. By $\tau_2$, we regard ${\mathbb{C}_M}^\times$ as the $\mathcal{S}pin^{\mathbb{C}}$ module and as the determinant module of $\mathcal{S}pin^{\mathbb{C}}$. By the Weil-Kostant theorem [Br p.66], we call $c \in \check{\mathrm{H}}^1(\mathbb{C}_M^\times)$ the first Chern class and it is isomorphic to the Picard group $\check{\mathrm{H}}^2(\mathbb{Z}_M)$.

For a local section $[\sigma_i, \lambda_i]$ of $\mathcal{S}pin^{\mathbb{C}}(n, \mathcal{C}_M^\omega)$ over $U_i$, we will express the local map over $U_{ij} \equiv U_i \cap U_j \neq$, $[\sigma_i, \lambda_i] = [\sigma_{ij}, \lambda_{ij}][\sigma_i, \lambda_i]$ where $[\sigma_{ij}, \lambda_{ij}] \equiv= [\sigma_i \sigma_j^{-1}, \lambda_i/\lambda_j]$. Then for $U_{ijk} \equiv U_i \cap U_j \cap U_k \neq$, we have the relation from the definition of $\mathcal{S}pin^{\mathbb{C}}$,

$$\sigma_{ij}\sigma_{jk}\sigma_{ki} = \lambda_{ij}\lambda_{jk}\lambda_{ki}. \tag{*}$$

Due to the relation $\tau(\sigma_{ij}\sigma_{jk}\sigma_{ki}) = 1$, $\sigma_{ij}\sigma_{jk}\sigma_{ki}$ must be an element of $\mathbb{Z}_2 = \{\pm 1\}$ valued second Čech cohomology, *i.e.*, element of Stiefel-Whitney class $w_2(M)$. Let $\epsilon_{ijk}$, $[\epsilon_{ijk}] \equiv w_2(M)$ be the value of above relation (*). The left hand side must be an element of the Picard group. We have the relation $w_2(M) = \alpha$ modulo $2\mathbb{Z}_M$.

The sufficiency is given as follows. We take the element $\alpha$ of the Picard group $\check{\mathrm{H}}^2(\mathbb{Z}_M)$ holding the relation in the propositions for all open set of $M$. Then we have a line bundle $LINE(M)$, whose first Chern class is $\alpha$, so that for an open set $U$ of $M$, its local trivialization is consist with $T^*M|_U = (SPIN^{\mathbb{C}}(M) \times_{\mathrm{SPIN}^{\mathbb{C}}(\mathbb{R}^n)} \mathbb{R}^n)|_U$ by choosing the relation (*). We can find the set $\{[\sigma_{ij}, \lambda]\}$ consisting with the relation (*) for all open set $U_{ijk}$. Then we have the spin $\mathbb{C}$ structure. $\square$

**Definition 3.14 (Spin Representation).** [G, Y]

*$M$ is a real $n$-dimensional compact manifold $M$ which has $\mathcal{S}pin^{\mathbb{C}}$ structure. Let an open set $U \subset M$,*

$$\Psi(U) = SPIN^{\mathbb{C}}(U) \times_{\mathrm{SPIN}^{\mathbb{C}}(\mathbb{R}^n)} \Psi(\mathbb{R}^n).$$

*Let $\Psi_M$ be a sheaf of analytic section of $\Psi(U)$ of $U \subset M$, which is called spinor sheaf.*

Then we have a relation $\Psi_M$ is sheaf isomorphic to $\mathbb{C}_M \otimes_{\mathbb{R}_M} \mathcal{C}liff_M$.

**Definition 3.15 (Levi-Civita connection of $\mathcal{S}pin^{\mathbb{C}}$).** [G, Y]

*Suppose that a real $n$-dimensional compact ringed manifold $(M, \mathcal{C}^{\omega}_M)$ has the Riemannian structure $(\Theta_M, \mathfrak{g}_M)$ and $\mathcal{S}pin^{\mathbb{C}}$ structure. If the connection of $\mathcal{S}pin^{\mathbb{C}}$ is induced from the the Levi-Civita connection $\nabla^{\mathrm{LC}}_M$ of $SO(n, \mathcal{C}^{\omega}_M)$, we will call it the Levi-Civita connection of $\mathcal{S}pin^{\mathbb{C}}$.*

**Proposition 3.16 (Levi-Civita connection of $\mathcal{S}pin^{\mathbb{C}}$).** [Y]

*The Levi-Civita connection $\nabla^{\mathrm{Spin}}_M$ of $\mathcal{S}pin^{\mathbb{C}}$ is the Clifford connection $\nabla^{\mathrm{Clf}}_M$.*

*Proof.* From the corollary 3.4 and the correspondence of $\tau_1$ in the proof of proposition 3.13, it is obvious. □

**Definition 3.17 (Dirac Operator).** [G, Y]

*The Dirac operator $\nabla\!\!\!\!/^{(\mathrm{Dir})}_M \in \mathcal{I}(\Psi)$ is defined by*

$$\nabla\!\!\!\!/^{\mathrm{Dir}}_M = \sum_{i=1}^{n} \mathfrak{c}(e^i) \cdot \nabla^{\mathrm{Spin}}_M{}_{\theta_i},$$

*where $< e^i, \theta_j >= \delta^i{}_j$. If we denote $\nabla^{\mathrm{Spin}}_M{}_{\partial_i}(e^j) = \sum_k w^j_{ik} e^k$,*

$$\nabla^{\mathrm{Spin}}_M{}_{\partial_i} = \partial_i + \frac{1}{4} \sum_{jk} w^j_{ik} \mathfrak{c}(e^k) \mathfrak{c}(e^j).$$

**Corollary 3.18 (Dirac Operator in $\mathbb{E}^n$).** [G, Y]

*The Dirac operator in $\mathbb{E}^n$ is given by*

$$\nabla\!\!\!\!/^{\mathrm{Dir}}_{\mathbb{E}^n} = \sum_{i=1}^{n} \mathfrak{c}(dx^i) \cdot \partial_i,$$

*where $x^i$ is the Cartesian coordinate.*

We sometimes use the notations $\partial\!\!\!/ := \nabla\!\!\!\!/^{\mathrm{Dir}}_M$ [Mat11, 16].

**Definition 3.19 (Dirac System).** [I-Bjo]

*We introduce a coherent $\mathcal{D}_M$-module for the Dirac equation, which is locally expressed by an exact sequence*

$$\mathcal{D}_M{}^{2^{[\frac{n}{2}]}} \xrightarrow{\sqrt{-1}\,\nabla\!\!\!\!/^{\mathrm{Dir}}} \mathcal{D}_M{}^{2^{[\frac{n}{2}]}} \xrightarrow{\eta} \nabla\!\!\!\!/\!\!\!-^{\mathrm{Dir}}_M \to 0.$$

*Let this $\mathcal{D}_M$-module $\nabla\!\!\!\!/\!\!\!-^{\mathrm{Dir}}_M$ be referred Dirac system in this article.*

Here for $\eta(\epsilon_i) = u_i$ for base $\epsilon_i := (0, \cdots, 0, 1, 0, \cdots 0) \in \mathcal{D}_M{}^{2^{[\frac{n}{2}]}}$, $\nabla\!\!\!\!/\!\!\!-^{\mathrm{Dir}}_M$ is expressed as $\nabla\!\!\!\!/\!\!\!-^{\mathrm{Dir}}_M = \sum \mathcal{D}_M u_i$. As the Dirac operator is acted by group action of $\mathcal{S}pin^{\mathbb{C}}$, $\mathcal{S}pin^{\mathbb{C}}$ acts on $\nabla\!\!\!\!/\!\!\!-^{\mathrm{Dir}}_M$.

As we argued in §3 in [I], we will also consider the (anti-)self-adjoint operators for the spinor representations module $\mathit{\Psi}_M$. We have the Hodge-star operator $*$ and it acts on $\mathit{\Psi}_M$

$$* : \mathit{\Psi}_M \to \mathit{\Psi}_M \otimes_{\mathcal{C}^\omega_M} w_M,$$

where $w_M$ is the volume form of $M$.

Let the local coordinate system $\{x^i\}_{1\le i\le n}$ have non-trivial Riemannian metric section $\mathfrak{g}_M = g_{M i,j} dx^i \otimes dx^j$, $g_M := \det g_{i,j}$ and $w_M = g_M dx^1 \cdots dx^n$. Then in general $\partial_i$ is not self-adjoint for $\sqrt{\mathcal{C}^\omega_M}$ because $* : \sqrt{\mathcal{C}^{\omega\,\mathbb{C}}_M} \to \sqrt{\mathcal{C}^{\omega\,\mathbb{C}}_M} w_M$ and adjoint form of $\partial_i$ also acts upon $w_M$. Thus we can define the anti-self-adjoint operator $\nabla^{\mathrm{SA}}_{M\,i} = g_M^{-1/4} \partial_i g_M^{1/4}$. Further we introduced a half form $\sqrt{\omega_M}$ in [I].

**Definition 3.20 (Spin Half Form & Spin Wave Sheaf).**

(1) *Let $\mathit{\Psi}_M \otimes_{\mathbb{R}_M} \sqrt{\omega_M}$ be spin half form. The local section of connection of spin half form $\mathcal{H}om_M(\Theta_M, \mathcal{I}(\mathit{\Psi}_M \otimes_{\mathbb{R}_M} \sqrt{\omega_M}))$ is expressed by ${\nabla^{\mathrm{SpinH}}_M}_\theta$ for $\theta \in \Theta_M(U)$:*

$${\nabla^{\mathrm{SpinH}}_M}_{\partial_i} = \partial_i - \frac{1}{4}\partial \log g_M + \frac{1}{4}\sum_{jk} w^j_{ik} \mathfrak{c}(e^k)\mathfrak{c}(e^j).$$

(2) *Let $\mathit{\Psi}_M \otimes_{\mathbb{R}_M} \sqrt{\mathcal{C}^{\omega\,\mathbb{C}}_M}$ be Spin wave sheaf. The local section of connection of spin half form $\mathcal{H}om_M(\Theta_M, \mathcal{I}(\mathit{\Psi}_M \otimes_{\mathbb{R}_M} \sqrt{\mathcal{C}^\omega_M}))$ is expressed by ${\nabla^{\mathrm{SpinW}}_M}_\theta$ for $\theta \in \Theta_M(U)$:*

$${\nabla^{\mathrm{SpinW}}_M}_{\partial_i} = \nabla^{\mathrm{SA}}_{M\,i} - \frac{1}{4}\partial \log g_M + \frac{1}{4}\sum_{jk} w^j_{ik} \mathfrak{c}(e^k)\mathfrak{c}(e^j).$$

*where $\nabla^{\mathrm{SA}}_{M\,\partial_i}$ is locally defined by $\nabla^{\mathrm{SA}}_{M\,i} = g_M^{-1/4} \partial_i g_M^{1/4}$ using the local coordinate system.*

**Proposition 3.21 (Spin Half Form).**

*The category whose object and morphism are given by $(\mathit{\Psi}_M \otimes_{\mathbb{R}_M} \sqrt{\omega_M}, {\nabla^{\mathrm{SpinH}}_M}_\theta)$ is category equivalent with that of $(\mathit{\Psi}_M \otimes_{\mathbb{R}_M} \sqrt{\mathcal{C}^\omega_M}, {\nabla^{\mathrm{SpinW}}_M}_\theta)$.*

*Proof.* This is essentially same as the proposition 2.5 in [I]. $\square$

**Remark 3.22.**

In fashion of [I], we should discriminate ${\nabla^{\mathrm{SpinW}}_M}_{\partial_i}$ and ${\nabla^{\mathrm{Spin}}_M}_{\partial_i}$ and explicitly write action of $\overline{\sigma}$ of the correspondence defined in the lemma I-3.24. However since ${\nabla^{\mathrm{SpinW}}_M}_{\partial_i} \equiv {\nabla^{\mathrm{Spin}}_M}_{\partial_i}$ in the primary meanings, we will neglect $\overline{\sigma}_M$ in this article for abbreviation.

## §4. Dirac Operators in a submanifold $S \hookrightarrow \mathbb{E}^n$

Let $S$ be a $k$-dimensional real analytic compact submanifold of $\mathbb{E}^n$, associated with an integrable connection ${\nabla^{\mathrm{SA}}_S}^{(s)}$, embedded in $\mathbb{E}^n$; $\iota_S : S \hookrightarrow \mathbb{E}^n$. Let $T_S$ a tubular neighborhood of $S$, $i_S : S \hookrightarrow T_S$ and $i_{T_S} : T_S \hookrightarrow \mathbb{E}^n$ such that $\iota_S \equiv i_{T_S} \circ i_S$. $T_S$ has a projection $\pi_{T_S} : T_S \to S$. $\mathbb{E}^n$ has a natural metric $\mathfrak{g}_{\mathbb{E}^n}$ and thus $T_S$ and $S$ has Riemannian module $(i^{-1}_{T_S}\Theta_{\mathbb{E}^n}, i^{-1}_{T_S}\mathfrak{g}_{\mathbb{E}^n})$ and $(\Omega_S, \iota^{-1}_S \mathfrak{g}_{\mathbb{E}^n})$.

**Notations 4.1.**

(1) *The inverse image of the Dirac system of* $\rlap{-}\forall^{\mathrm{Dir}}_{T_S}$ *is defined as*

$$\rlap{-}\forall^{\mathrm{Dir}}_{S\to T_S} = \mathcal{D}_S \otimes_{i_S^{-1}\mathcal{D}_{T_S}} i_S^{-1}(\rlap{-}\forall^{\mathrm{Dir}}_{T_S}),$$

*where*

$$\rlap{-}\forall^{\mathrm{Dir}}_{T_S} = \mathcal{D}_{T_S} \otimes_{i_{T_S}^{-1}\mathcal{D}_{\mathbb{E}^n}} i_{T_S}^{-1}(\rlap{-}\forall^{\mathrm{Dir}}_{\mathbb{E}^n}).$$

(2) *The tangent sheaf of* $\Theta_{T_S}$ *is* $\Theta_{T_S} = i_{T_S}^{-1}\Theta_{\mathbb{E}^n}$ *and has a direct decomposition as a* $\mathcal{C}^{\omega}_{T_S}$*-module,*

$$\Theta_{T_S} = \Theta^{\|}_{T_S} \oplus \Theta^{\perp}_{T_S},$$

*where* $i_S^{-1}\Theta^{\|}_{T_S} = \Theta^{\|}_S$, $i_S^{-1}\Theta^{\perp}_{T_S} = \Theta^{\perp}_S$ *and* $\Theta^{\perp}_S$ *is the normal sheaf defined by the exact sequence,*

$$0 \to \Theta_S \to \iota_S^{-1}\Theta_M \to \Theta^{\perp}_S \to 0.$$

(3) *An affine vector (coordinate)* $\mathbf{Y} \equiv (Y^i)$ *in* $T_S \subset \mathbb{E}^n$ *is expressed by,*

$$\mathbf{Y} = \mathbf{X} + \mathbf{e}_{\dot\alpha} q^{\dot\alpha},$$

*for a certain affine vector* $\mathbf{X}$ *of* $S$*.*

(4) *A point* $p$ *in* $T_S$ *is expressed by the local coordinate* $(u^{\mu}) := (s^1, s^2, \cdots, s^k, q^{k+1}, \cdots, q^n)$, $\mu = 1, 2, \cdots, n$ *where* $(s^1, \cdots, s^k)$ *is a local coordinate of* $\pi_{T_S} p$*; We assume that the beginning of the Greek* ($\alpha$, $\beta$, $\gamma$, $\cdots$) *runs from 1 to k and they with dot* ($\dot\alpha$, $\dot\beta$, $\dot\gamma$, $\cdots$) *runs form* $k+1$ *to* $n$*). Let* $(u^{\mu}) = (s^{\alpha}, q^{\dot\alpha})$*, where the ending of the Greek* ($\mu$, $\nu$, $\lambda$, $\cdots$) *run from* 1 *to* $n$*.*

(5) $\mathbf{E}_{\mu} := \partial_{\mu} := \partial/\partial u^{\mu}$ *is a base of* $\Theta_{T_S}(U)$*.* $\mathbf{E}^{\mu} := du^{\mu}$ *is a base of* $\Omega^1_{T_S}(U)$*:* $<\mathbf{E}^{\mu}, \mathbf{E}_{\nu}>= \delta^{\mu}_{\nu}$

(6) *For* $U \subset T_S$*, the induced metric of* $T_S$ *from* $\mathbb{E}^n$ *has a direct sum form,*

$$\mathfrak{g}_{T_S} := i_{T_S}^{-1}\mathfrak{g}_{\mathbb{E}^n} = \mathfrak{g}_{T_S^{\|}} \oplus \mathfrak{g}_{T_S^{\perp}},$$

*where* $\mathfrak{g}_{T_S^{\perp}}$ *has a trivial structure. In local coordinate,*

$$\mathfrak{g}_{T_S^{\perp}} = \delta_{\dot\alpha,\dot\beta} \mathrm{d}q^{\dot\alpha} \otimes \mathrm{d}q^{\dot\beta}, \quad \mathfrak{g}_{T_S^{\|}} = g_{T_S\alpha\beta} \mathrm{d}s^{\alpha} \otimes \mathrm{d}s^{\beta},$$

*or for* $g_{T_S\mu,\nu} := \mathfrak{g}_{T_S}(\partial_{\mu}, \partial_{\nu})$*,*

$$g_{T_S\dot\alpha\dot\beta} = \delta_{\dot\alpha\dot\beta}, \quad g_{T_S\dot\alpha\beta} = g_{T_S\alpha\dot\beta} = 0,$$

*where* $\partial_{\mu} := \partial/\partial u^{\mu}$*.*

**Proposition 4.2.**

*Let the anti-self-adjoint connection* $\nabla^{\mathrm{SA}}_{S\ \dot\alpha} \in \Gamma(U, i_S^*\Xi_{T_S}(\Theta^{\perp}_{T_S}))$ *for an open set* $U$ *in* $S$*. There is an injective endomorphism of the Dirac system* $\rlap{-}\forall_{S\to T_S}$*,*

$$\eta^{\mathrm{conf}}_{\dot\alpha} : \rlap{-}\forall^{\mathrm{Dir}}_{S\to T_S} \to \rlap{-}\forall^{\mathrm{Dir}}_{S\to T_S},$$

*for* $P \in \Gamma(U, \rlap{-}\forall^{\mathrm{Dir}}_{S\to T_S})$*,* $\eta^{\mathrm{conf}}_{\dot\alpha}(P) = P\nabla^{\mathrm{SA}}_{S\ \dot\alpha} \in \Gamma(U, \rlap{-}\forall^{\mathrm{Dir}}_{S\to T_S})$*. Then we have a submodule of* $\rlap{-}\forall^{\mathrm{Dir}}_{S\to T_S}$*,*

$$\eta^{\mathrm{conf}} : (\overline{\mathbb{S}}_{S\to T_S})^{n-k} \to \sum_{\dot\alpha=k+1}^{n} \rlap{-}\forall^{\mathrm{Dir}}_{S\to T_S} \nabla^{\mathrm{SA}}_{S\ \dot\alpha} \subset \rlap{-}\forall^{\mathrm{Dir}}_{S\to T_S}.$$

*Proof.* For the tubular neighborhood, we have a local coordinate system as we demonstrated in [I]. Using such local coordinate system, this can be proved as we did in proposition I-4.1. $\square$

**Definition 4.3.**

(1) *We will define a coherent $\mathcal{D}_S$-module, $\not{\nabla}^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$, by the exact sequence,*

$$(\not{\nabla}^{\mathrm{Dir}}_{S\to T_S})^{n-k} \xrightarrow{\sigma^{\mathrm{conf}}} \not{\nabla}^{\mathrm{Dir}}_{S\to T_S} \longrightarrow \not{\nabla}^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n} \longrightarrow 0,$$

*Let us call it submanifold Dirac system of $S \hookrightarrow \mathbb{E}^n$.*

(2) *Let the submanifold Dirac system $\not{\nabla}^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ be decomposed by the exact sequence,*

$$\mathcal{D}_S^{2^{[\frac{n}{2}]}} \xrightarrow{\sqrt{-1}\,\not{\nabla}_{S\hookrightarrow\mathbb{E}^n}} \mathcal{D}_S^{2^{[\frac{n}{2}]}} \longrightarrow \not{\nabla}^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n} \to 0.$$

*where we tune $\not{\nabla}_{S\hookrightarrow\mathbb{E}^n}$ such that there is a map $\sigma : \mathcal{C}liff_S \to i_S^{-1}\mathcal{C}liff_{T_S}$*

$$[\sigma(\not{\nabla}_S) - \not{\nabla}_{S\hookrightarrow\mathbb{E}^n}] \in \mathcal{C}_S^{\omega} \otimes i_S^{-1}\mathcal{C}liff_{T_S}.$$

*we will call $\not{\nabla}_{S\hookrightarrow\mathbb{E}^n}$ is the the submanifold Dirac operator of $S \hookrightarrow \mathbb{E}^n$.*

We will describe our main theorem, which is given in [Mat1-3, 5, 6, 10, 11, 16].

**Theorem 4.4.**

(1) *$k = 1$ and $n = 3$ case, $S$ is curve $C$,*

$$\not{\nabla}^{\mathrm{Dir}}_{C\hookrightarrow\mathbb{E}^2} = \begin{pmatrix} \partial_s & \frac{1}{2}\kappa_{\mathbb{C}} \\ -\frac{1}{2}\overline{\kappa_{\mathbb{C}}} & \partial_s \end{pmatrix},$$

*where $s$ is the arclength of the curve $C$, $\kappa_{\mathbb{C}}$ is the complex curvature of $C$, defined by $\kappa_{\mathbb{C}} = \kappa(s)\exp\left(\sqrt{-1}\int^s \tau ds\right)$ using the Frenet-Serret curvature $\kappa$ and torsion $\tau$.*

(2) *$k = 2$ and $n = 3$ case, $S$ is a conformal surface,*

$$\not{\nabla}^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^2} = 2\rho^{1/2}\begin{pmatrix} p & \partial \\ \overline{\partial} & p \end{pmatrix}\rho^{-1/2},$$

*where $(z, \bar{z})$ is a complex parameterization of $S$, $p = \rho^{1/2}H/2$ when the volume element of $S$ is given by $dvol = \rho dz d\bar{z}$ and $H$ is the mean curvature.*

(3) *$k = 2$ and $n = 4$ case, $S$ is a conformal surface,*

$$\not{\nabla}^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^4} = 2\rho^{1/2}\begin{pmatrix} & & p_c & \partial \\ & & \overline{\partial} & -\overline{p}_c \\ \overline{p}_c & \partial & & \\ \overline{\partial} & -p_c & & \end{pmatrix}\rho^{-1/2},$$

*where $(z, \bar{z})$ is a complex parameterization of $S$, $p_c = \rho^{1/2}H_c$ when the volume element of $S$ is $dvol = \rho dz d\bar{z}$ and $H_c$ is "complex" mean curvature (see example 4.14 (3) in [I]).*

**Remark 4.5.**

(1) The $(k,n)=(1,2)$ case is studied by me. When the affine vector of $C$ ($i_C : C \hookrightarrow \mathbb{E}^2$) is expressed by $X = X_1 + \sqrt{-1}X_2$, we have a relation,

$$\mathrm{Hom}_{\mathbb{C}}(\not\nabla^{\mathrm{Dir}}_{C\hookrightarrow\mathbb{E}^2}, i_C^{-1}C^{\omega}_{\mathbb{E}^2}) = \mathbb{C}\begin{pmatrix} \sqrt{\partial_s X} \\ \sqrt{-\partial_s X}\end{pmatrix}.$$

This is essentially the same as the Frenet-Serret relation.

(2) The case that $(k,n)=(2,3)$ and $S$ is a conformal surface is identified with the generalized Weierstrass relation [Mat11] whose form was found by Konopelchenko and Taimanov [Ko1, 2, KT, T1, 2]. (Kenmotsu found more naive expression of the generalized Weierstrass relation [Ke]). Let $\mathbb{E}^3 \approx \mathbb{C}\times\mathbb{E} \ni (Z := X_1 + \sqrt{-1}X_2, X^3)$, the solution of the Dirac equation is given as,

$$\mathrm{Hom}_{\mathbb{C}}(\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^3}, i_S^{-1}C^{\omega}_{\mathbb{E}^3}) = \mathbb{C}\begin{pmatrix} \sqrt{\partial_s Z} \\ \sqrt{-\partial_s Z}\end{pmatrix}.$$

(3) Konopelchenko found the generalized Weierstrass relation as the zero mode of the Dirac operator $\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^3}$ through the studies of geometrical interpretation of higher dimensional soliton. Bobenko also pointed out that the Dirac operator plays an important role in geometrical interpretation of soliton equation [Bo]. It known that Abresh already found similar equation ending of 1980's on the investigation of constant mean curvature surface.

In [Mat2], I conjectured that the submanifold Dirac operator might exhibit submanifold feature and be connected to higher dimensional soliton as open problems. Berguress and Jenssen followed my approach [TM] and calculated the Dirac operator of a surface immersed in $\mathbb{E}^3$ [BJ]. However their surface is not conformal, it was too complex for me to obtain any meaningful result. As I read [KT], I got the above result [Mat11] by means of more physical method.

On the other hand, Pedit and Pinkall developed the quaternion algebraic geometry and reformulated the Dirac operator on a conformal surface in $\mathbb{E}^3$ in framework of quaternion algebraic geometry [Pe,PP]. Friedrich also gave a the Dirac operator of a surface immersed in $\mathbb{E}^3$ through the study of spinor bundle [Fr]. He gave a spinor connection of a submanifold and his approach is very resemble to our methods. However it is not clear why one considers the Dirac operator since his approach can not connect with a submanifold Schrödinger operator. We must emphasize that submanifold differential operators are universal and not a special feature of the spinor bundle. In fact, in optics or sound wave, we encounter similar differential effects [Mi]. Accordingly as we described in the §1 introduction,the Dirac operator $\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ should be studied with the Schrödinger operator $\Delta^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ and so on.

Further I also gave more concrete conjecture on the Dirac operator on a conformal surface in $\mathbb{E}^4$ in ending of 1997: the Dirac operator $\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^4}$ is the operator of the generalized Weierstrass relation of a conformal surface in $\mathbb{E}^4$ [Mat16]. At the same time, Konopelchenko [Ko2] and Pedit and Pinkall [PP] investigated the generalized Weierstrass relation of a conformal surface in $\mathbb{E}^4$ and found the Dirac operator $\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^4}$. In other words, as things turned out, my conjecture was proved.

Here after we will prove the theorem 4.4.

**Lemma 4.6.**

(1) *A element of left-hand $\mathcal{D}_S$-module $P \in \not\nabla^{\mathrm{Dir}}_{S\to T_S}$ is also given by $Q \in \not\nabla^{\mathrm{Dir}}_{T_S}$ such that $P := Q|_{q^{\dot\alpha}\equiv 0, \dot\alpha=k+1,\cdots,n)})$.*

(2) *An element of left-hand $\mathcal{D}_S$-module $P \in \not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ is given by $Q \in \not\nabla^{\mathrm{Dir}}_{S\to T_S}$ such that $P := Q|_{\nabla^{\mathrm{SA}}_{S\ \dot\alpha}\equiv 0,(\dot\alpha=k+1,\cdots,n)}$.*

(3) *The Dirac operator $\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ is unique up to the action of $\mathcal{S}pin^{\mathbb{C}}$.*

*Proof.* As did in proposition I-4.6, we can prove them. $\square$

**Lemma 4.7.**

(1) *The Dirac operator $\not\nabla^{\mathrm{Dir}}_{T_S} \equiv i^{-1}_{T_S}\not\nabla^{\mathrm{Dir}}_{\mathbb{E}^n}$, which is expressed by affine coordinate $\not\nabla^{\mathrm{Dir}}_{T_S} = \mathfrak{c}(dX^i)\cdot\nabla^{\mathrm{Spin}}_{T_S\partial_i}$, can be written by a local coordinate system,*

$$\not\nabla^{\mathrm{Dir}}_{T_S} = \mathfrak{c}(E^\mu)\cdot\nabla^{\mathrm{Spin}}_{T_S\ \ \mu},$$

*where*

$$\nabla^{\mathrm{Spin}}_{T_S\ \ \mu} \equiv \nabla^{\mathrm{Spin}}_{T_S\ \ E_\mu} = \partial_\mu + \Omega_{T_S\,\mu},$$

$\partial_\mu \equiv E_\mu$*,* $du^\mu \equiv E^\mu$*,* $\Omega_{T_S\,\mu} = \frac{1}{4}\Omega_{T_S\,\mu\nu}^{\ \ \ \ \lambda}\mathfrak{c}(E^\mu)\mathfrak{c}(E_\lambda)$ *and* $\nabla^{\mathrm{LC}}_{T_S\,\mu}E_\nu = \Omega_{T_S\,\mu\nu}^{\ \ \ \ \lambda}E_\lambda$.

*Proof.* Direct computations leads the above results. $\square$

**Lemma 4.8.**

(1)

$$\nabla^{\mathrm{Spin}}_{T_S\ \ \alpha} = \pi^*_{T_S}\nabla^{\mathrm{SA}}_{S\ \ \alpha} + \Omega_{T_S\,\alpha} - \frac{1}{4}\partial_\alpha \log g_{T_S}.$$

(2)

$$\nabla^{\mathrm{Spin}}_{T_S\ \ \dot\alpha} = \nabla^{\mathrm{SA}}_{T_S\,E_{\dot\alpha}} - \frac{1}{4}\partial_{\dot\alpha}\log g_{T_S}.$$

(3) *For $k = 2$ case,*

$$-\frac{1}{4}\partial_{\dot\alpha}\log g_{T_S} = \frac{H_{\dot\alpha-2} - \partial_{\dot\alpha}J(q^{\dot\beta})/2}{1 - 2H_{\dot\alpha}q^{\dot\alpha} + J(q^{\dot\beta})}.$$

*Proof.* Using the expressions in propositions and lemmas 4.5-10 in [I], above results are obtained [BJ, Mat11, 16]. $\square$

**Lemma 4.9.**

(1) *When we define $\not\nabla^{\mathrm{Dir}}_{S\to T_S} := \not\nabla^{\mathrm{Dir}}_{T_S^{\parallel}}|_{q^{\dot\alpha}\equiv 0}$, $\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ is expressed as $\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n} = \not\nabla^{\mathrm{Dir}}_{S\to T_S}|_{\nabla^{\mathrm{SA}}_{T_S\,E_{\dot\alpha}}\equiv 0}$.*

(2) *For $k = 2$ case, $\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ is expressed by $\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n} = \not\nabla^{\mathrm{Dir}}_{S} + \mathfrak{c}(E^{\dot\alpha})H_{\dot\alpha}$.*

*Proof.* From the definition 4.3, (1) is obvious. Using (1) and lemma 4.7 (3), (2) is obtained. $\square$

For a while, we will deal with only $n = 4$ and $k = 2$ case.

**Lemma 4.10.** [Po, Mat16]

*Let* $(n,k)=(4,2)$ *and* $S$ *be a conformal surface and* $\rho := g_S^{1/2}$*. A natural euclidean inner space coordinate system is expressed by* $y^a, y^b, \cdots$, $(a,b=1,2)$*. The moving frames of* $S$ *is given by* $ds^\alpha \equiv e^\alpha$, $\partial_\alpha \equiv e_\alpha$*.*

(1) *The moving frame is written as,*

$$e^a{}_\alpha = \rho^{1/2}\delta^a{}_\alpha.$$

(2) *The Christoffel symbol over a conformal surface* $S$ *is calculated as,*

$$\gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}\rho^{-1}(\delta^\alpha{}_\beta \partial_\gamma \rho + \delta^\alpha{}_\gamma \partial_\beta \rho - \delta_{\beta\gamma}\partial_\alpha \rho).$$

(3) *The spin connection over* $S$ $\omega_\alpha := \Omega_\alpha|_{q^{\dot\alpha}\equiv 0}$ *becomes*

$$\omega_\alpha = -\frac{1}{4}\rho^{-1}\sigma^{ab}(\partial_a\rho\delta_{\alpha b} - \partial_b\rho\delta_{\alpha a}),$$

*where* $\sigma^{ab} := 1\otimes[\sigma^a,\sigma^b]/2$*.*

(4) *The Dirac operator* $\not\nabla_S^{\mathrm{Dir}}$ *can be expressed as*

$$\not\nabla_S^{\mathrm{Dir}} = \sigma^1\otimes\sigma^a\delta^\alpha{}_a[\rho^{-1/2}\partial_\alpha + \frac{1}{2}\rho^{-3/2}(\partial_\alpha\rho)].$$

(5) *The anti-self-adjoint operator of* $S$ *is given by*

$$\nabla^{\mathrm{SA}}_{S\ \alpha} = \rho^{-1/2}\partial_\alpha\rho.$$

(6) *The normal Clifford elements can be connected to the Pauli matrices* $\sigma^a$*,*

$$\mathfrak{c}(e^\alpha) = e_a{}^\alpha\sigma^1\otimes_{\mathbb{C}}\sigma^a, \quad \mathfrak{c}(e^{\dot\alpha}) = \sigma^2\otimes_{\mathbb{C}}\sigma^{\dot\alpha-2}.$$

*Proof.* See the p.232 in [Po] or direct computations also give (1)-(5). (6) was obtained in [Mat11,16] by direct computations along the definition of 4.3 □

**Corollary 4.11.**

$\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^4}$ *can be expressed by*

$$\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^4} = \rho^{-1}[\sigma^1\otimes\sigma^a\delta^\alpha{}_a\partial_\alpha + \rho^{1/2}\sigma^1\otimes\sigma^a H_a]\rho^{1/2}.$$

**Lemma 4.12.**

(1) *Let* $S\subset\mathbb{E}^3_s\times\mathbb{E}^1_t\equiv\mathbb{E}^4$ *be a product manifold* $C\times\mathbb{E}^1_t$ *such that for* $\pi_{\mathbb{E}^4\to\mathbb{E}^3_s}:\mathbb{E}^4\to\mathbb{E}^3_s$*,* $\pi_{\mathbb{E}^4\to\mathbb{E}^3_s}(S)$ *is a space curve* $C$ *in* $\mathbb{E}^3$*. We set* $s$ *be an arclength of* $C$ *and then we obtain the relation,*

$$H_1 = -\frac{1}{2}tr_2(\gamma^1{}_{31}), \quad H_2 = -\frac{1}{2}tr_2(\gamma^1{}_{41}), \quad \rho = 1.$$

(2) *Let consider the case that the surface* S *immersed in* $\mathbb{E}^3$ *and the direction* $\mathbf{e}_4$ *is globally identified with the direction of* $X^4$*. It means that* $H_2=0$ *and* $p_c$ *is obtained as,*

$$p_c \equiv p := \frac{1}{2}\sqrt{\rho}H_1.$$

*Proof.* They are obvious □

**Proof of Theorem 4.4.**

Corollary 4.11 and lemma 4.12 gives a complete proof of theorem 4.4. $\square$

Here we will mention the results in [Mat2, 6, 14]. Though we will not give proofs in this article, following results were also obtained by physical computations.

**Proposition 4.13.**

(1) *$S$ is one dimensional case, $\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ is identified with the Ferret-Serret Relation and one of Lax operators of the generalized MKdV equations .*

(2) *$S$ is a conformal surface and $n = 3, 4$ case, $\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ is identified with the generalized Weierstrass relation and related to generalized MNV equation .*

*Comment.* (1) was obtained in [Mat1-3, MT] and (2) was proved by another approach in [Ko1,2, KT, T1,T2].

**Proposition 4.14.**

*Let us consider a localized ring $\mathcal{C}^{\omega\mathbb{C}}_S[[\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}]]$ and consider $\det(\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n})$.*

(1) *$S$ is one dimensional case, $\log\det\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ is identified with the Euler-Bernoulli functional $\mathcal{B} = \int \mathrm{d}s|k|^2$.*

(2) *$S$ is one dimensional case, $\mathrm{index}\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ gives the fundamental group of the loop .*

(3) *$S$ is a conformal surface case, $\log\det\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ is related to the Willmore functional $\mathcal{W} = \int \mathrm{d}z\mathrm{d}\bar{z}\rho H^2$ and area $\mathcal{A}$ of the surface.*

*Comment.* (1) was obtained in [Mat2, 6, 8], (2) was in [Mat2, 6, 8] and (3) was in [Mat14]. (1) and (3) were proved by path integral method and heat kernel method. (3) was calculated using the path integral and generalized Hurwitz $\zeta$-regularzation.

**Remark 4.15.**

For the case of $n = 3$ dimensional case, 4.14 (3) means that

$$\mathcal{W} = \mu^2\mathcal{A} + \frac{5}{3}\pi\chi - \nu - \log\det\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n},$$

where $\mu$ and $\nu$ are certain real numbers and $\chi$ is Euler number. Hence the Willmore conjecture should be studied in the framework of this Dirac operator [Mat14].

**Conjecture 4.16.**

*For any manifold $M$ and $S$, the Dirac operator can represents the submanifold system.*

**Remark 4.17.**

We should note that differential algebra sheaf $\mathcal{D}_S$ generated by $\nabla^{\mathrm{SA}}_{S\ \theta}$ $(\theta \in \Theta_S)$ commutes with that $\mathcal{D}^{\perp}_S$ generated by $\nabla^{\mathrm{SA}}_{S\ \theta}$ $(\theta \in \Theta^{\perp}_S)$. In the algebra $\mathcal{D}_{S\to\mathbb{E}^n}$, these are commutant if we used terminology in von-Neumann algebra. Then $S$ itself might be regarded as $\mathcal{D}_{S\to\mathbb{E}^n} \cap \Theta_S$ or factor of the system. Hence above behavior might be very natural in non-commutative algebra theory.

**Remark 4.18.**

Since our Dirac operator represents the submanifold completely, at least, surface and curve case, we can naturally determine the operator form of $\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ by data of geometry. We will comment upon this correspondence from categorical point of view.

We first consider a category of analytic submanifold in $\mathbb{E}^n$. We refer it $Submfd$, whose object is an analytic submanifold $S \hookrightarrow \mathbb{E}^n$. For two submanifolds $S \hookrightarrow \mathbb{E}^n$ and $S' \hookrightarrow \mathbb{E}^n$, there is an analytic map $f$ between them as a morphism in the category *Submfd*. For these submanifolds, we naturally define a correspondence between $\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ and $\not\nabla^{\mathrm{Dir}}_{S'\hookrightarrow\mathbb{E}^n}$. We will introduce a category of Dirac operator $Dirac_{\cdot\hookrightarrow\mathbb{E}^n}$ whose object is the formal infinite series ring $\mathcal{C}^{\omega\mathbb{C}}_S[[\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}]]$ and morphism $f^*$ is induced from $f$. Then, we can find an equivalent functor $\sigma$ between the category *Submfd* of submanifolds in $\mathbb{E}^n$ and $Dirac_{\cdot\hookrightarrow\mathbb{E}^n}$; morphism is given as $f$.

On the other hand, we can consider an analytic category *Anal* defined over parameter space $(s^1, \cdots, s^k, t)$, whose object is a subset of formal power series $\mathbb{C}[[\partial_\alpha, s^\alpha, t, \partial_t]]$. We can naturally define a forgetful functor $\nu_{Anal}$ from the $Dirac_{\cdot\hookrightarrow\mathbb{E}^n}$ to *Anal*. Then it is natural to consider a $\mathbb{C}$ vector space,

$$\mathcal{A}^{\mathrm{Dir}}_{\cdot\hookrightarrow\mathbb{E}^n}[\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}] := \{A \in \mathbb{C}[[\partial_\alpha, s^\alpha]] \subset \mathbb{C}[[\partial_\alpha, s^\alpha, t, \partial_t]] \mid [\partial_t - A, \not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}] = 0\ \}/\mathcal{C}^{\omega\mathbb{C}}_S[[\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}]].$$

To determine $\mathcal{A}^{\mathrm{Dir}}_{\cdot\hookrightarrow\mathbb{E}^n}$ is commutant problem as that in von-Neumann algebra. The intersection of commutants $\mathcal{A}^{\mathrm{Dir}}_{\cdot\hookrightarrow\mathbb{E}^n}[\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}] \cap \mathcal{C}^{\omega\mathbb{C}}_S[[\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}]]$ is related to a commutative ring. For the cases of curves and conformal surfaces, the intersection is associated with algebraic curve *e.g.*, a hyperelliptic curve $\mathrm{Spec}(\mathbb{C}[x,y]/(y^2 - f(x))$ . As it might, guess, be a pun of someone, $\mathrm{Spect}(\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n})$ corresponds to $\mathrm{Spec}(\mathbb{C}[x,y]/(y^2 - f(x))$. Former is of a noncommutative ring defined over differential geometrical manifold and is expressed by a differential operator associated with $\mathrm{K}^0$-group in K-theory. Latter is of commutative ring defined over algebraic variety and is expressed by a commutative variable associated with $\mathrm{K}^1$-group in the K-theory [Con]. Using the orbits, we can introduce linear topology and classify $\mathcal{A}^{\mathrm{Dir}}_{\cdot\hookrightarrow\mathbb{E}^n}[\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}]$ itself. Investigations of these sets have been already done in the framework of soliton theory. Geometrical interpretation of soliton theory can be regarded as a composite functor from *Anal* to *Submfd*.

The morphism $f$ in *Submfd* induces a morphism $f^\#$ in *Anal*. Inverse of this functor means that if we could classify the operators set $\mathcal{A}^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}[\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}]$'s, we can classify submanifolds in *Submfd*. For example, let us consider a morphism $f^\#$ such that $\mathcal{A}^{\mathrm{Dir}}_{\cdot\hookrightarrow\mathbb{E}^n}[\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}]$ and $\mathcal{A}'^{\mathrm{Dir}}_{\cdot\hookrightarrow\mathbb{E}^n}[\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}]$ are connected by an orbit of $t$ and we denote it $f^\#_t$. Its inverse functor of $f^\#_t$ implies that there is a deformation $f_t$ of a submanifolds $S$ and $S'$. Since the Dirac operator $\not\nabla^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ connects between them, we emphasize that the Dirac operator itself may behave as an interpreter like a functor.

In these cases, by using $\mathcal{A}^{\mathrm{Dir}}_{S\hookrightarrow\mathbb{E}^n}$ we can classify geometrical objects or its moduli and can investigate an (algebraic) relation between classified objects [Mat12, 13, 15, 17]. Moreover physically speaking, such classification and investigation of their relations can be interpreted as a quantization of submanifold geometry as we showed in [Mat12, 13, 15, 17].

As mentioned in the introduction, one of our purposes of this study is arthmeticalization of geometry. In the case of soliton theory, there is an infinite dimensional Lie group acting the $\mathcal{A}^{\mathrm{Dir}}_{\cdot\hookrightarrow\mathbb{E}^n}$ and it governs the structure of objects of *Anal*. As Klein and Lie wished, above program might enable us to investigate *Submfd*. In other words, this is a Galois theory of submanifold. There the Dirac operator plays the same role as $\sqrt{-1}$ in algebraic number theory. It implies that even though invariances in the differential geometry and harmonic map theory are usually given by quadratic form, I believe that we must factorize or construct a "Dirac operator" and adjoin such a object; I consider that such a way is a way to unify mathematics from discovery of $\sqrt{-1}$.

From physical point of view, as I showed in [Mat12], quantized submanifold contains various submanifolds and is governed by such a large group. If the quantized submanifold is given first, there naturally appears the Dirac operator and Dirac field (as an object that defines a dimensionality of functional space or geometry). This might give us a deep philosophical question what is an origin of material or fermion.

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